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LOS ALAMOS SCIENTIFIC LABORATORY
OF THE UNIVERSITY OF CALIFORNIA ○ LOS ALAMOS NEW MEXICO

COORDINATE TRANSFORMATIONS IN INTRANUCLEAR CASCADE STUDIES

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ABSTRACT

The spatial and Lorentz transformations employed in the treatment of the collision of two relativistic particles are presented. The form chosen is particularly suitable to numerical calculation and has been used in the intranuclear cascade studies made on Maniac I and II.

INTRODUCTION

We consider the transformations involved in a collision between two relativistic particles, labeled 1 and 2, giving rise to two or more particles 3,4,5,... Such transformations are of interest in high energy nuclear phenomena and have been used in recent calculations on intranuclear cascades.(1)

Two types of collisions are distinguished (neglecting polarization effects):

(i) elastic collisions; the incoming pair of particles identical with the two outgoing particles.

(ii) inelastic collisions; either the incoming and the outgoing pair are different, or there are more than two emerging particles, i.e., particle production has occurred in the collision.

The incoming particles 1 and 2 are moving in arbitrary directions with respect to the laboratory system of coordinates XYZ, whose origin is conveniently taken at the point of collision.

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1. Monte Carlo Calculations on Intranuclear Cascades. Phys. Rev. 110. 185 (1958).

Let $\underline{\eta}_1$, $\underline{\eta}_2$ be the relativistic momenta in XYZ of the two incoming particles with rest masses m_1, m_2 and relativistic energies γ_1, γ_2. $\underline{\eta}_i$ $(i = 1,2)$ is the momentum in units of $m_i c$; γ_i is the relativistic energy in units of $m_i c^2$, where c is the velocity of light. The configuration is shown in Fig. 1.

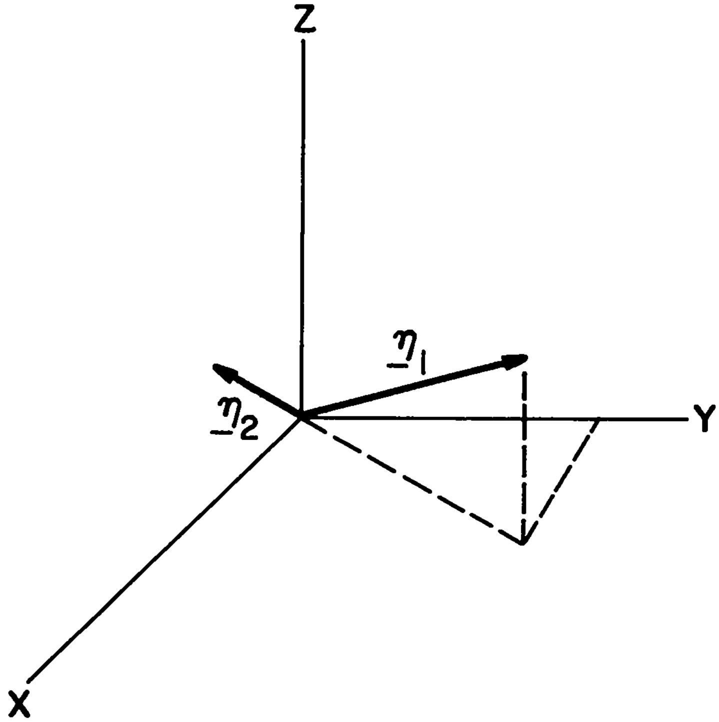

Fig. 1. Momentum vectors of two colliding particles in the laboratory frame of reference XYZ.

The over-all objective is to find the Lorentz transformation, which operates on the two 4-vectors[2] $(\underline{\eta}_1, \gamma_1)$ and $(\underline{\eta}_2, \gamma_2)$, and

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2. For definiteness, we regard all vectors as column vectors, e.g., $(\underline{\eta}_1, \gamma_1) = \begin{pmatrix} \eta_{1x} \\ \eta_{1y} \\ \eta_{1z} \\ \gamma_1 \end{pmatrix}$.

yields transformed vectors $(\underline{\eta}_{1c}, \gamma_{1c})$, $(\underline{\eta}_{2c}, \gamma_{2c})$ in the center-of-mass system of the two particles. The result of the collision is to produce a new set of 4-vectors $(\underline{\eta}_{jc}, \gamma_{jc})$ with $j = 3,4,5,\ldots$; the particular set depends on the nature of the collision. The new vectors are then referred to the laboratory system XYZ by means of a Lorentz transformation, which is the inverse of the original one.

Alternatively, one may first perform a series of rotations of the original laboratory system XYZ in such a fashion as to simplify the Lorentz transformation. As before, after the collision the inverse Lorentz transformation (also much simpler) is applied; this is followed by the series of inverse rotations that finally describe the new vectors in XYZ. We choose to follow this alternative proposal.

PRELIMINARY ROTATIONS

Referring to Fig. 1, one may describe the first objective as the rotation of XYZ such that

(i) the new X-axis is along $\underline{\eta}_1$,

(ii) the new Z-axis lies in the plane of $\underline{\eta}_1$, $\underline{\eta}_2$.

Indeed it is convenient to regard this first rotation as composed of two steps that accomplish (i) and (ii) in succession. In the nuclear cascade studies, it turns out that the azimuthal orientation of $\underline{\eta}_2$ about the $\underline{\eta}_1$-axis has a uniform distribution and leads to a (computationally) simple rotation matrix for step (ii).

Let $\alpha_{11}, \alpha_{12}, \alpha_{13}$ be the direction cosines of $\underline{\eta}_1$ with respect to the X-, Y-, Z-axes, respectively. There is a wide variety of rotations possible to accomplish step (i). A relatively simple one is the following: consider $\underline{\eta}_1$ being oriented at some longitude and latitude angles with respect to the X-axis regarded as $(0^\circ, 0^\circ)$. A rotation about the polar axis corresponding to the longitudinal angle of $\underline{\eta}_1$, followed by a rotation about a horizontal axis through the latitudinal angle, effects step (i). Written explicitly, the matrix is

$$R \equiv \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \frac{-\alpha_{12}}{\sqrt{1-\alpha_{13}^2}} & \frac{\alpha_{11}}{\sqrt{1-\alpha_{13}^2}} & 0 \\ \frac{-\alpha_{11}\alpha_{13}}{\sqrt{1-\alpha_{13}^2}} & \frac{-\alpha_{12}\alpha_{13}}{\sqrt{1-\alpha_{13}^2}} & \sqrt{1-\alpha_{13}^2} \end{pmatrix} \qquad (1)$$

If η_1 represents the magnitude of $\underline{\eta}_1$, then

$$R\underline{\eta}_1 = R \begin{bmatrix} \eta_{1x} \\ \eta_{1y} \\ \eta_{1z} \end{bmatrix} = R \begin{bmatrix} \eta_1 \alpha_{11} \\ \eta_1 \alpha_{12} \\ \eta_1 \alpha_{13} \end{bmatrix} \Longrightarrow \begin{bmatrix} \eta_1 \\ 0 \\ 0 \end{bmatrix} \qquad (2)$$

as expected.

Step (ii) is a simple rotation about $\underline{\eta}_1$, the direction of the new X-axis, through an angle called ϕ , such that the Z-axis now lies in the plane[3] of $\underline{\eta}_1$, $\underline{\eta}_2$. In the intranuclear cascade

3. For ease in keeping track of signs, the choice of the two possible values of ϕ is such that the Z component (in the new system) of $\underline{\eta}_2$ is non-negative. The range of ϕ is thus $0 \rightarrow 2\pi$.

studies, ϕ is a random variable uniformly distributed in the interval $0 \rightarrow 2\pi$. (In general, if $\underline{\eta}_2$ is some prescribed vector, then ϕ would be calculated from the direction cosines of $\underline{\eta}_1$, $\underline{\eta}_2$.) Hence the appropriate rotation matrix is

$$\Phi \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \tag{3}$$

If $X_1Y_1Z_1$ denote the new coordinate system, the orientation of $\underline{\eta}_1$ and $\underline{\eta}_2$ is as shown in Fig. 2. If μ is the cosine of the

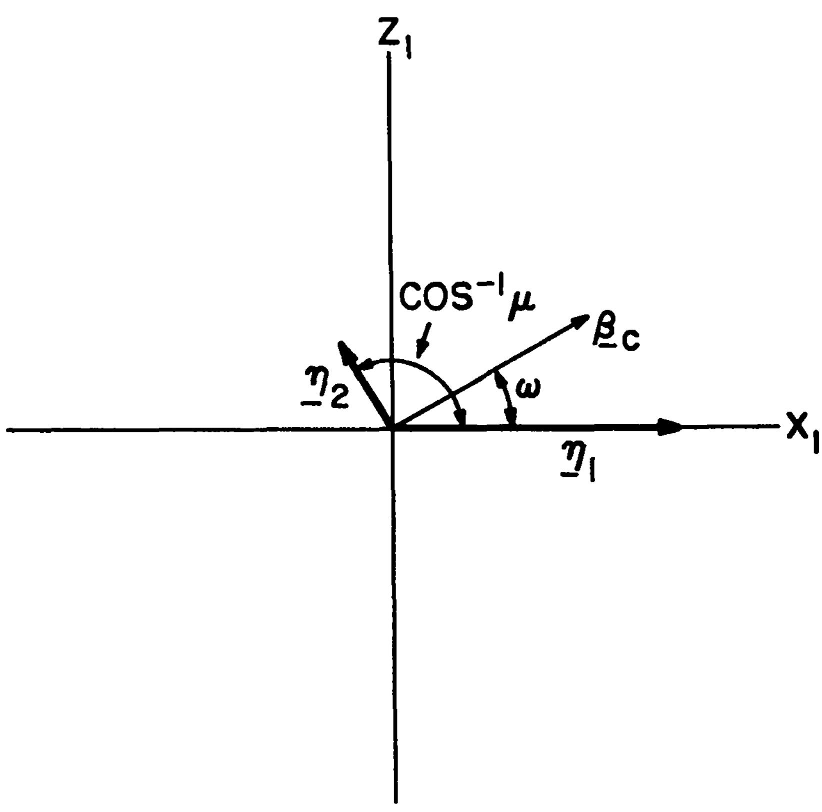

Fig. 2. The rotated frame of reference, showing the X_1 Z_1 plane, which contains both $\underline{\eta}_1$ and $\underline{\eta}_2$.

enclosed angle between $\underline{\eta}_1$, $\underline{\eta}_2$, then the components in $X_1Y_1Z_1$ may be written

$$\underline{\eta}_1 = \begin{bmatrix} \eta_1 \\ 0 \\ 0 \end{bmatrix} \qquad \underline{\eta}_2 = \begin{bmatrix} \eta_2\mu \\ 0 \\ \eta_2\sqrt{1-\mu^2} \end{bmatrix} \tag{4}$$

Since the transformations are pure rotations, γ_1, γ_2 remain unaltered. It is thus seen that the components of the incoming momenta have a relatively simple form in the new coordinate system; they represent the starting point in the nuclear cascade calculation, i.e., the transformations R followed by Φ need not be performed explicitly. However, after the collision, the vectors (resulting from the collision) are eventually subjected to Φ^{-1} and R^{-1} in that sequence tc express them in the laboratory system XYZ.

Thus far we have arranged the rotation of XYZ so that the incoming motion of the colliding particles is contained in the new X_1Z_1 plane. However, we are not quite ready for the Lorentz system. One more rotation is desired and this stems from the fact that the center-of-mass motion is not along the X_1-axis. The simplest Lorentz transformation follows whenever that circumstance does obtain. As we shall see momentarily the desired rotation (of Fig. 2) is that of the X_1Z_1 plane about the Y_1-axis by an angle called ω, where

$$\cos\omega = \frac{\underline{\beta}_c \cdot \underline{\eta}_1}{\beta_c \eta_1} \tag{5}$$

where $\underline{\beta}_c$ is the velocity of the center-of-mass in units of c, and is given by

$$\underline{\beta}_c = \frac{m_1\underline{\eta}_1 + m_2\underline{\eta}_2}{m_1\gamma_1 + m_2\gamma_2} \qquad (6)$$

a well-known result from the theory of relativity.

Before developing Eq. (5) further, we first define a few auxiliary quantities:

$$\begin{aligned} \Gamma &= m_1\gamma_1 + m_2\gamma_2 \\ A &= \gamma_1\gamma_2 - \eta_1\eta_2\mu \\ N^2 &= m_1^2 + m_2^2 + 2m_1m_2A \\ \underline{H} &= m_1\underline{\eta}_1 + m_2\underline{\eta}_2 \end{aligned} \qquad (7)$$

Using the relation $\gamma_i^2 = 1 + \eta_i^2$ $(i = 1,2)$, it can be shown that

$$\underline{H}\cdot\underline{H} = H^2 = \Gamma^2 - N^2 \qquad (8)$$

Γ is the total energy of the two-body system in the laboratory frame XYZ. A physical interpretation for A is that it represents the total energy of particle 1 in the rest frame of particle 2. N is the total energy in the center-of-mass system, and $\underline{H}$ is the total momentum in XYZ.

Equation 5 can be written as

$$\cos\omega = \frac{m_1\eta_1 + m_2\eta_2\mu}{H} \qquad (9)$$

and the rotation about Y_1-axis by the angle ω puts the X-axis along the direction of the center-of-mass $\underline{\beta}_c$. Explicitly,

$$\Omega \equiv \begin{pmatrix} \cos\omega & 0 & \sin\omega \\ 0 & 1 & 0 \\ -\sin\omega & 0 & \cos\omega \end{pmatrix} \tag{10}$$

LORENTZ TRANSFORMATION

The transformation to the center-of-mass system $X_cY_cZ_c$ involves a Lorentz transformation L, i.e., symbolically,

$$\begin{array}{ccc} & X_1Y_1Z_1 & & X_cY_cZ_c \\ L\Omega & \begin{bmatrix} \eta_1 \\ 0 \\ 0 \\ \gamma_1 \end{bmatrix} & \Longrightarrow & \begin{bmatrix} \eta_{1xc} \\ 0 \\ \eta_{1zc} \\ \gamma_{1c} \end{bmatrix} \end{array}$$

and (11)

$$L\Omega \begin{bmatrix} \eta_2\mu \\ 0 \\ \eta_2\sqrt{1-\mu^2} \\ \gamma_2 \end{bmatrix} \Longrightarrow \begin{bmatrix} \eta_{2xc} \\ 0 \\ \eta_{2zc} \\ \gamma_{2c} \end{bmatrix}$$

As is well known the momenta of particles 1 and 2 are equal in magnitude but opposite in direction, i.e., $m_1\underline{\eta}_{1c} = -m_2\underline{\eta}_{2c}$. Figure 3 is a typical example of the state of affairs before the collision, in the center-of-mass frame; δ is the angle between $\underline{\eta}_{1c}$ and $\underline{\beta}_c$, (the latter lies along the X-axis in the center-of-

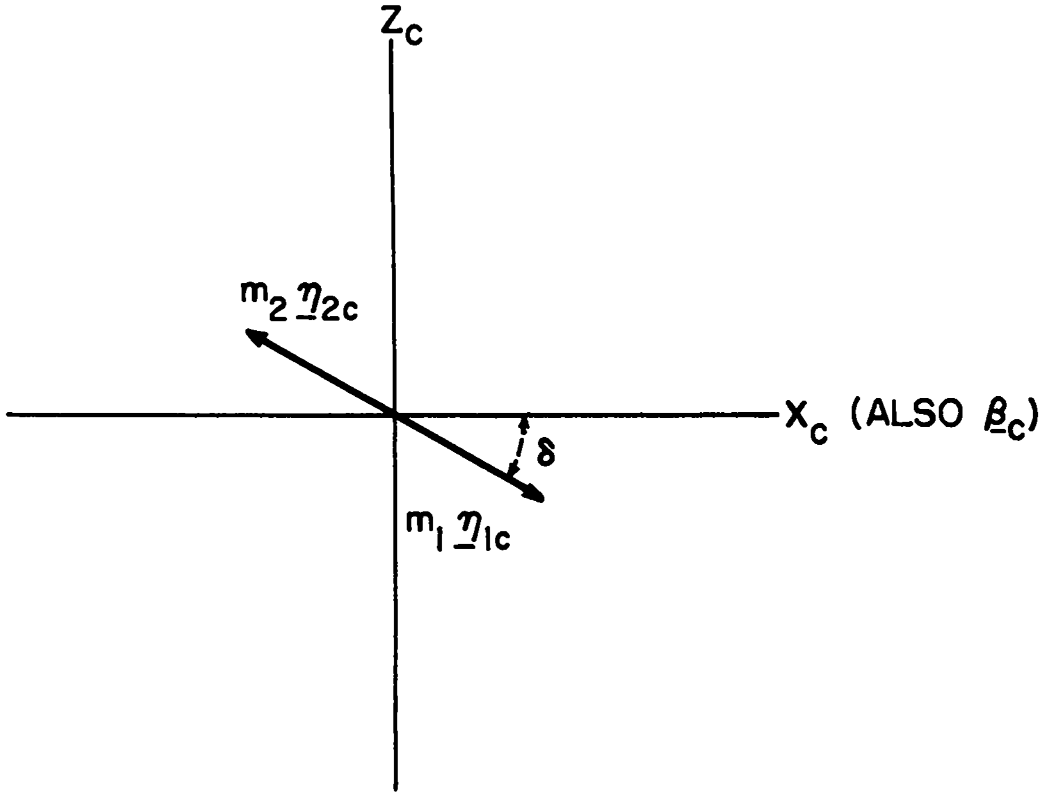

Fig. 3. Center-of-mass system. The X-axis is along $\underline{\beta}_c$, the velocity vector of the center-of-mass frame. $m_1\underline{\eta}_{1c}$ is equal in magnitude but opposite in direction to $m_2\underline{\eta}_{2c}$. δ is the angle subtended by $\underline{\eta}_1$.

mass frame). Again from the theory of relativity, one can show that

$$L = \frac{1}{N}\begin{pmatrix} \Gamma & 0 & 0 & -H \\ 0 & N & 0 & 0 \\ 0 & 0 & N & 0 \\ -H & 0 & 0 & \Gamma \end{pmatrix} \qquad (12)$$

with Γ, H, N as defined in Equation 7.

In view of subsequent rotations that may occur in the collision process itself, it is convenient to express the momentum components on the right in Equation 11 in terms of the magnitude of the momentum and the angle δ. For example, in terms of these quantities η_{1xc} in Equation 11 is given by $\eta_{1xc} = \eta_{1c} \cos \delta$. Furthermore, we can express these quantities, as well as the total energies, in terms of A, N, H (cf. Equation 7). After some algebraic manipulation, we may write for the magnitude of the momenta in the center-of-mass frame:

$$\eta_{1c} = \frac{m_2 \sqrt{A^2-1}}{N} \qquad \eta_{2c} = \frac{m_1 \sqrt{A^2-1}}{N} \tag{13}$$

and for the corresponding energies

$$\gamma_{1c} = \frac{m_1 + m_2 A}{N} \qquad \gamma_{2c} = \frac{m_2 + m_1 A}{N} \tag{14}$$

and finally,

$$\cos \delta = \frac{\gamma_1(m_1 A + m_2) - \gamma_2(m_1 + m_2 A)}{H \sqrt{A^2-1}} \tag{15}$$

We point out that δ is in the range $-\pi \leq \delta \leq 0$, hence $\sin \delta \leq 0$ always, i.e.,

$$\sin \delta = - \sqrt{1-\cos^2 \delta}$$

One can see that the negative sign is appropriate as follows: $\underline{\beta}_c$ is related to the vector sum of $\underline{\eta}_1$, $\underline{\eta}_2$; hence it lies in the region included by them. Therefore ω, the angle between $\underline{\eta}_1$ and $\underline{\beta}_c$ is positive (cf. Figure 2). Evaluating η_{1zc} in Equation (11), one finds $\eta_{1zc} = -\eta_1 \sin \omega \leq 0$; therefore $\delta \leq 0$.

Summarizing, we may rewrite the right hand side of Equation 11:

$$\begin{bmatrix} \eta_{1xc} \\ 0 \\ \eta_{1zc} \\ \gamma_{1c} \end{bmatrix} = \frac{1}{N} \begin{bmatrix} m_2 \sqrt{A^2-1} \cos\delta \\ 0 \\ m_2 \sqrt{A^2-1} \sin\delta \\ m_1 + m_2 A \end{bmatrix}$$

$$\begin{bmatrix} \eta_{2xc} \\ 0 \\ \eta_{2zc} \\ \gamma_{2c} \end{bmatrix} = \frac{1}{N} \begin{bmatrix} -m_1 \sqrt{A^2-1} \cos\delta \\ 0 \\ -m_1 \sqrt{A^2-1} \sin\delta \\ m_2 + m_1 A \end{bmatrix} \qquad (16)$$

THE COLLISION

The two 4-vectors of Equation 16 are the input data involved in the determination of the nature of the collision. The effect of the collision is to produce a new set of vectors in the center-of-mass frame. In order to express these vectors in the laboratory system XYZ, one subjects them to the transformations L^{-1}, Ω^{-1}, Φ^{-1}, finally R^{-1}, in that order. As mentioned earlier, if particle production occurs the set contains more than two members; in any case the new vectors need not lie in the original plane of the motion, in fact they may not be co-planar. We complete the discussion for the case where the physical process involved in the collision is a simple elastic scattering.

ELASTIC SCATTERING

In this instance the new vectors of the collision are simply the old vectors $\underline{\eta}_{1c}$, $\underline{\eta}_{2c}$ rotated with two degrees of freedom. The first of these may be considered as a rotation X in the plane of $X_c Z_c$ (i.e., the plane of $\underline{\eta}_1$, $\underline{\eta}_2$), followed by a second rotation about the collision axis (along $\underline{\eta}_{1c}$). This latter angle of rotation is called θ and (if the particles are not polarized) is uniformly distributed in the range $(0 \longrightarrow \pi)$. It turns out to be simpler computationally to restrict X to

$$0 \leq X \leq \pi$$

and extend θ to

$$0 \leq \theta \leq 2\pi$$

The effect of the rotation through an angle X may be accomplished by the following substitutions in the expressions for the vector components of the colliding particles:

$$\cos (X - |\delta|) \longrightarrow \cos \delta$$

$$\sin (X - |\delta|) \longrightarrow \sin \delta$$

where the absolute value is used to remind one that $\delta \leq 0$.

The rotation (about $\underline{\eta}_{1c}$) of the plane $\underline{\eta}_1$, $\underline{\eta}_2$ requires several transformations. In the first place, the center-of-mass frame must be rotated so that the new X-axis is along $\underline{\eta}_{1c}$. This angle is, however, precisely the angle δ (≤ 0). This rotation is succeeded by a rotation of the vectors[4] by an angle, θ, about the collision axis (along $\underline{\eta}_{1c}$). Finally, in order to restore the status quo, the inverse of the center-of-mass rotation is made. This triplet of transformations may be written

$$\Delta^{-1}\,\Theta\,\Delta$$

where

$$\Delta = \begin{pmatrix} \cos\delta & 0 & \sin\delta \\ 0 & 1 & 0 \\ -\sin\delta & 0 & \cos\delta \end{pmatrix} \qquad (17)$$

and

$$\Theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \qquad (18)$$

and

$$\Delta^{-1} = \begin{pmatrix} \cos\delta & 0 & -\sin\delta \\ 0 & 1 & 0 \\ \sin\delta & 0 & \cos\delta \end{pmatrix} \qquad (19)$$

INVERSE TRANSFORMATIONS

When the new vectors resulting from the collision process have been established, we begin to apply a series of transformations that express the vectors in the original laboratory frame XYZ.

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4. This is the only occasion where the vectors themselves are rotated. In all other transformations, it is the coordinate system that is rotated.

The first of the inverse transformations is the Lorentz transformation

$$L^{-1} = \frac{1}{N}\begin{pmatrix} \Gamma & 0 & 0 & H \\ 0 & N & 0 & 0 \\ 0 & 0 & N & 0 \\ H & 0 & 0 & \Gamma \end{pmatrix} \tag{20}$$

This is followed in turn by

$$\Omega^{-1} = \begin{pmatrix} \cos\omega & 0 & -\sin\omega \\ 0 & 1 & 0 \\ \sin\omega & 0 & \cos\omega \end{pmatrix} \tag{21}$$

then,

$$\Phi^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} \tag{22}$$

and finally,

$$R^{-1} = \begin{pmatrix} \alpha_{11} & \frac{-\alpha_{12}}{\sqrt{1-\alpha_{13}^2}} & \frac{-\alpha_{11}\alpha_{13}}{\sqrt{1-\alpha_{13}^2}} \\ \alpha_{12} & \frac{\alpha_{11}}{\sqrt{1-\alpha_{13}^2}} & \frac{-\alpha_{12}\alpha_{13}}{\sqrt{1-\alpha_{13}^2}} \\ \alpha_{13} & 0 & \sqrt{1-\alpha_{13}^2} \end{pmatrix} \tag{23}$$

We have taken the trouble to exhibit the last three matrices in order to be quite explicit and complete; the brief statement, that inverse rotations are transposes of the original matrices, might have sufficed.

All the matrices are well behaved, except possibly for R and its inverse R^{-1}. If $\underline{\eta}_1$ is oriented very nearly parallel to the Z-axis in the laboratory frame of reference, the radical in the denominator of some of the R matrix elements approaches zero and may give rise to computational difficulties. It should be observed that no formal complications are encountered since the limits are well behaved as $\sqrt{1-\alpha_{13}^2} \longrightarrow 0$.

A reasonable computational procedure for this case is the following: determine whether α_{13} is within some very small region, say 10^{-5}, of unity. If it is, take $\alpha_{12} = 0$, let $\alpha_{11} \longrightarrow 0$ as $\alpha_{13} \longrightarrow 1$. In the limit R becomes

$$R_{limit} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad (24)$$

with R^{-1}_{limit} equal to the transpose of R_{limit}.